Unit Root Tests in R
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According to Pfaff (2008), "A discrete time series is defined as an ordered sequence of random numbers with respect to time." However, some time series exhibit trending behaviors and are non-stationary, such as Fig. 1. Stationarity is about the stability of the data over time and the statistical properties of the data are invariant over time. Stationarity means that the mean, variance, and autocovariance are time-independent.

If the data is trending (a persistent long-term movement), the researcher needs to remove the trend by using the first difference and time-trend regression. The first difference, ($y_t - y_{t-1}$), is appropriate for a time series integrated of order 1, $I(1)$. Time series and time-trend regression are appropriate for trend stationary $I(0)$ time series. When two variables are trending over time, regressing one on the other could have a high coefficient of determination (R^2) even when they are unrelated.

There are several unit root tests, such as Dickey-Fuller (DF), Augmented Dickey-Fuller (ADF), Phillips-Perron (PP), Kwiatkowski–Phillips–Schmidt–Shin (KPSS), Elliott, Rothenberg and Stock (ERS), and others. Baum (2009) argued that the standard DF, ADF, and PP tests have low power to reject the null hypothesis that the series is non-stationary, $I(1)$ rather than stationary, $I(0)$.

Say we are looking at the United States of America's (USA) gross domestic product (GDP) between the first quarter of 1967 and the first quarter of 2024, Fig. 1.

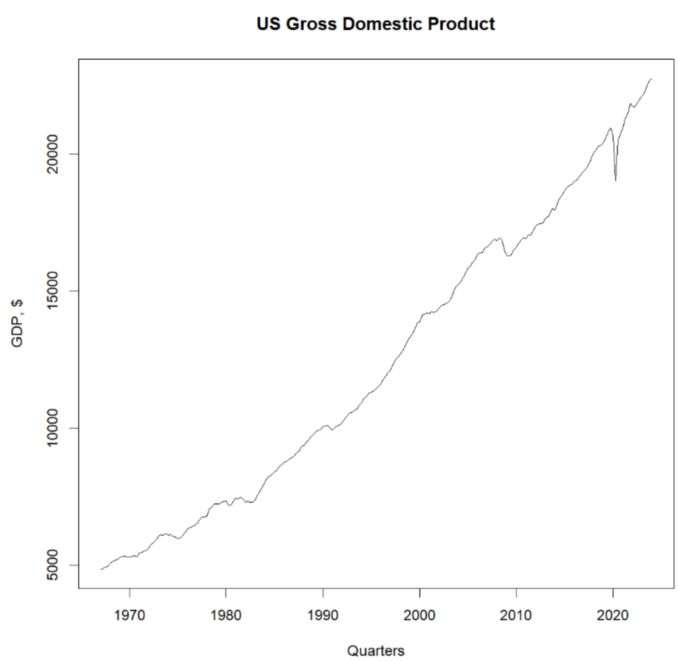

Figure 1 – US Gross Domestic Product

The Dickey-Fuller Unit Root Test

Baun (2009) noted that the standard Dickey-Fuller test is essentially an ordinary least square (OLS). To determine if a time series has a unit root, the researcher must perform unit root tests. A unit root renders the time series nonstationary when the value of alpha (α) = 1 in the equation below.

$$y_t = \alpha y_{t-1} + \beta x_t + \varepsilon_t \qquad (1)$$

Subtracting y_{t-1}, from each side of the equation, results in an equivalent equation $\Delta y_t = \gamma y_{t-1} + \varepsilon_t$, where gamma ($\gamma$) = ($\alpha - 1$). Thus, if $\alpha = 1$ then $\gamma = 0$.

There are three different regression equations that Dickey-Fuller (1979) used to test for the presence of unit root:

$$\Delta y_t = \gamma y_{t-1} + \varepsilon_t \qquad (2)$$

$$\Delta y_t = \alpha_0 + \gamma y_{t-1} + \varepsilon_t \qquad (3)$$

$$\Delta y_t = \alpha_0 + \gamma y_{t-1} + \alpha_2 t + \varepsilon_t \qquad (4)$$

The null hypothesis (H_0) is that the time series is non-stationary, it has a unit root. The alternative hypothesis (H_A) is that the time series is stationary.

H_0: $\alpha = 1$ (differencing makes the data stationary)

H_A: $\alpha < 1$ (the data is stationary)

Dickey and Fuller used OLS regression to estimate the parameter γ and the standard error (SE). If $\gamma = 0$, then y_t contains a unit root. Comparing the t-statistic, γ/SE, with the appropriate values in the Dickey-Fuller tables allows the researcher to accept or reject the null hypothesis that $\gamma = 0$.

If the t-statistic is greater than the critical value, we do not reject the null hypothesis.

Augmented Dickey-Fuller Unit Root Test

The ADF unit root test includes lag changes and testing the hypotheses that $\gamma = 0$ uses the same critical values. The autoregressive process equations are:

$$\Delta y_t = \gamma y_{t-1} + \sum_{i=2}^{p} \beta_i \Delta y_{t-i+1} + \varepsilon_t \qquad (5)$$

$$\Delta y_t = \alpha_0 + \gamma y_{t-1} + \sum_{i=2}^{p} \beta_i \Delta y_{t-i+1} + \varepsilon_t \qquad (6)$$

$$\Delta y_t = \alpha_0 + \gamma y_{t-1} + \alpha_2 t + \sum_{i=2}^{p} \beta_i \Delta y_{t-i+1} + \varepsilon_t \qquad (7)$$

Pfaff (2008) noted, "The ADF test has been implemented in the contributed R packages fUnitRoots, tseries, urca, and uroot as functions adftest(), adf.test(), ur.df(), and ADF.test(), respectively."

We use the URCA package (Pfaff) function ur.df() to compute the augmented Dickey-Fuller (ADF) unit root test for determining if a time series is nonstationary or stationary. The null hypothesis for the ADF test tests is that $\alpha = 1$ in the following model equation.

$$y_t = c + \beta t + \alpha y_{t-1} + \varphi \Delta y_{t-1} + \varepsilon_t$$

The null hypothesis (H_0) is that the time series is nonstationary, it has a unit root. The alternative hypothesis (H_A) is that the time series is stationary.

H_0: $\alpha = 1$ (the data is nonstationary)

H_A: $\alpha < 1$ (the data is stationary)

The key point is that the null hypothesis assumes that $\alpha = 1$, hence the p-value should be less than the value at the significance level at 5 percent to reject the null hypothesis and accept that the time series is stationary.

Below is the URCA code for ADF in R:

```
ur.df(y, type = c("none", "drift", "trend"), lags = 1, selectlags = c("Fixed", "AIC", "BIC"))
```

y is a vector; type is either "none," "drift," or "trend;" lags are the number of lags of the endogenous variable; and "selectlags" can be achieved by using the Akaike "AIC," or the Bayes "BIC" information criteria. The maximum number of lags considered is set by lags. The default is to use a "fixed" lag length set by lags.

When the type is "none," the regression test does not include the intercept or the trend. When set to "drift," the intercept is included, and when set to "trend" the regression includes both the intercept and trend.

If the test statistic is less than the critical value at a chosen significance level, 5 percent, we reject the null hypothesis and conclude that the stock price series is stationary. Conversely, if the test statistic is more, we fail to reject the null hypothesis, suggesting non-stationarity.

Code in R:

```
> summary(ur.df(gdp.ts[,"Y"], type = c("trend"), lags = 1, selectlags =
("AIC")))
```

```
###############################################
# Augmented Dickey-Fuller Test Unit Root Test #
###############################################

Test regression trend

Call:
lm(formula = z.diff ~ z.lag.1 + 1 + tt + z.diff.lag)

Residuals:
    Min      1Q  Median      3Q     Max
-1783.53  -38.57    9.28   52.46 1072.90
```

Coefficients:

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	148.57583	59.37677	2.502	0.0131 *
z.lag.1	-0.03155	0.01709	-1.846	0.0662.
tt	2.88459	1.36532	2.113	0.0357 *
z.diff.lag	-0.14638	0.06606	-2.216	0.0277 *

Signif. codes: 0 ***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 168.4 on 223 degrees of freedom
Multiple R-squared: 0.05569, Adjusted R-squared: 0.04299
F-statistic: 4.384 on 3 and 223 DF, p-value: 0.005073

Value of test-statistics is: -1.8465, 20.6473, 4.1918

Critical values for test statistics:

	1pct	5pct	10pct
tau3	-3.99	-3.43	-3.13
phi2	6.22	4.75	4.07
phi3	8.43	6.49	5.47

The test statistic is -1.8465 which is more than the critical value tau3 at the 5 percent level of significance. Likewise, the p-value is greater than the 5 percent critical value. Therefore, we do not reject the null hypothesis and conclude that the series is nonstationary.

Given that the data is nonstationary, we differentiate the data and perform another ADF unit root test. The result is shown below. It indicates that the test statistic is less than the critical value at the 5 percent level of significance. So, the differenced data is stationary.

```
> macro.df2=(ur.df(diff(gdp.ts[,"Y"]), type = c("trend"), lags = 1, selectlags
= c("AIC")))
> summary(macro.df2)

###############################################
# Augmented Dickey-Fuller Test Unit Root Test #
###############################################

Test regression trend


Call:
lm(formula = z.diff ~ z.lag.1 + 1 + tt + z.diff.lag)

Residuals:
     Min       1Q    Median      3Q      Max
   1808.35   -35.80    9.15    54.15   1040.17

Coefficients:
```

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	50.81507	23.17403	2.193	0.0294 *
z.lag.1	-1.23822	0.10205	-12.134	<2e-16 ***
tt	0.40904	0.17632	2.32	0.0213 *
z.diff.lag	0.06663	0.06699	0.995	0.3210

```
Signif. codes: 0 ***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 '' 1

Residual standard error: 169.7 on 222 degrees of freedom
Multiple R-squared: 0.5823,    Adjusted R-squared: 0.5766
F-statistic: 103.1 on 3 and 222 DF, p-value: < 2.2e-16
```

```
The value of the test statistic is: -12.1341 49.0797 73.6181
Critical values for test statistics:
```

	1pct	5pct	10pct
tau3	-3.99	-3.43	-3.13
phi2	6.22	4.75	4.07
phi3	8.43	6.49	5.47

The test statistic for tau is -12.1341, which is smaller than the critical value at the 5 percent level of significance. Therefore, we reject the null hypothesis of a unit root and conclude that the differentiated data is stationary.

Fig. 2 shows the residuals, autocorrelations, and the partial autocorrelations of the residuals.

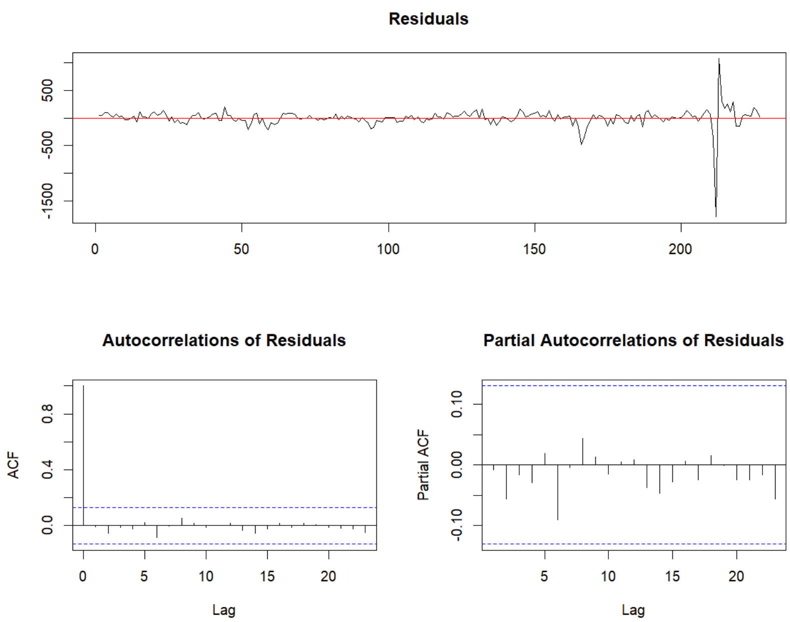

Figure 2 – ADF Test Fit and Residual Diagnosis

Phillips-Perron (PP) Test

The PP test addresses some of the limitations of the ADF test by accounting for autocorrelation and heteroscedasticity. It compares the value of the test statistic to the Dickey-Fuller critical values. Phillips and Perron (1998) used non-parametric test statistics for the null hypothesis.

The null hypothesis (H_0) is that the time series is non-stationary, it has a unit root. The alternative hypothesis (H_A) is that the time series is stationary.

H_0: $\alpha = 1$ (the data is nonstationary)

H_A: $\alpha < 1$ (the data is stationary)

Below is the URCA code for the PP test in R:

ur.pp(x, type = c("Z-alpha", "Z-tau"), model = c("constant", "trend"), lags = c("short", "long"), use.lag = NULL)

If the test statistic is greater than the critical value at the 5 percent level of significance, we do not reject the null hypothesis of a unit root and conclude that the series is nonstationary.

Below is the URCA code for PP in R:

```
> summary(ur.pp(gdp.ts[,"Y"], type = ("Z-tau"), model = c("trend"), use.lag =
1))
```

####################################

Phillips-Perron Unit Root Test

####################################

Test regression with intercept and trend

Call:

lm(formula = y ~ y.l1 + trend)

Residuals:

Min	1Q	Median	3Q	Max
-1723.09	-39.23	6.85	51.56	1323.08

Coefficients:

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	518.00179	207.64707	2.495	0.0133 *
y.l1	0.96422	0.01687	57.154	<2e-16 ***
trend	3.16894	1.34778	2.351	0.0196 *

Signif. codes: 0 ***' 0.001 **' 0.01 '*' 0.05 '.' 0.1 '' 1

Residual standard error: 169.5 on 225 degrees of freedom
Multiple R-squared: 0.999, Adjusted R-squared: 0.999
F-statistic: 1.105e+05 on 2 and 225 DF, p-value: < 2.2e-16

Value of test-statistic, type: Z-tau is: -1.9955

	aux. Z statistics
Z-tau-mu	2.9872
Z-tau-beta	2.2415

Critical values for Z statistics:

	1pct	5pct	10pct
critical values	-4.001349	-3.430658	-3.138651

Elliott-Rothenberg-Stock (ERS) DF-GLS Unit Root Test

Elliott-Rothenberg-Stock (1996) used a different test statistic and critical values. "The DF-GLS test makes use of generalized least squares (GLS) rather than OLS and has been shown to have considerably higher power in many circumstances (Baum, 2009)."

According to Baum, "The KPSS test may be used to confirm the findings of a DF-GLS test; their verdicts will not necessarily agree, but if they do, that is strong evidence in favor of (non-)stationarity."

H_0: $\alpha = 1$: (the null hypothesis is that a unit root is present in an autoregressive time series model)

H_A: $\alpha < 1$ (a unit root is not present in an autoregressive time series model)

Below is the URCA code for ERS in R:

Usage ur.ers(y, type = c("DF-GLS", "P-test"), model = c("constant", "trend"), lag.max = 4)

If the test statistics is greater than the 5 percent confidence value, we reject the null hypothesis.???

In the "ur.ers" root test within the URCA package in R, you reject the null hypothesis when the calculated test statistic is significantly lower than the critical value at your chosen significance level

```
> summary(ur.ers(gdp.ts[,"Y"], type = c("DF-GLS"), model = c("trend"), lag.max
= 1))

###############################################
# Elliot, Rothenberg, and Stock Unit Root Test #
###############################################

Test of type DF-GLS
detrending of series with intercept and trend


Call:
lm(formula = dfgls.form, data = data.dfgls)

Residuals:
     Min       1Q   Median       3Q      Max
-1747.13   -41.70    10.33    58.95  1165.17

Coefficients:
```

	Estimate	Std. Error	t value	Pr(>|t|)
yd.lag	-0.007623	0.012272	-0.621	0.5351
yd.diff.lag1	-0.135586	0.066447	-2.041	0.0425 *

```
Signif. codes: 0 ***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 '' 1

Residual standard error: 170.8 on 225 degrees of freedom
Multiple R-squared: 0.02151,   Adjusted R-squared: 0.01281
F-statistic: 2.473 on 2 and 225 DF, p-value: 0.08664


The value of the test statistic is: -0.6212

Critical values of DF-GLS are:
```

	1pct	5pct	10pct
critical values	-3.48	-2.89	-2.57

Kwiatkowski-Phillips-Schmidt-Shin Test

The KPSS test assumes the null hypothesis of stationarity against the alternative of a unit root, which is the opposite of the ADF, PP, and ERS tests. The KPSS test compares the computed test statistic against critical values obtained from the KPSS distribution table.

H_0: The null hypothesis for the test is that the data is stationary.

H_A: The alternate hypothesis for the test is that the data is nonstationary.

Thus, the interpretation of the test statistic and the p-value is opposite to the ADF, PP, and ERS tests.

Below is the URCA code for KPSS in R:

ur.kpss(y, type = c("mu", "tau"), lags = c("short", "long", "nil"), use.lag = NULL)

If the test statistic exceeds the critical value, the null hypothesis of stationarity is rejected.

```
#######################################
# KPSS Unit Root / Cointegration Test #
#######################################

The value of the test statistic is: 11.468

> summary(ur.kpss(gdp.ts[,"Y"], type = c("mu"), use.lag = 1))

#######################
# KPSS Unit Root Test #
#######################

The test is of type: mu with 1 lag.

The value of the test statistic is: 11.468

Critical value for a significance level of:
```

	10pct	5pct	2.5pct	1pct
critical values	0.347	0.463	0.574	0.739

References

Baum, C. F. (2009). Tests for stationarity and stability in the time series. Boston College.

Dickey, D.A. and Fuller, W.A. (1979). Distribution of the Estimators for Autoregressive Time Series with a Unit Root. Journal of the American Statistical Association, 47, 427-431.

Elliott, G., Rothenberg, T. J., and Stock, J. H. (1996). Efficient Tests for an Autoregressive Unit Root. Econometrica, 64:4, 813-836.

Pfaff, B. (2008). Analysis of Integrated and Cointegrated Time series in R. Springer Science+Business Media, LLC.

Phillips, P. C. and Perron, P. (1988). Testing for a Unit Root in Time Series Regression, Vol. 75, No. 2 (Jun. 1988), pp. 335-346. Biometrika.